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CLAW-FREENESS, 3-HOMOGENEOUS SUBSETS OF A GRAPH AND A RECONSTRUCTION PROBLEM

MAURICE POUZET, HAMZA SI KADDOUR, AND NICOLAS TROTIGNON

Abstract. We describe Forb$\{K_{1,3}, \overline{K_{1,3}}\}$, the class of graphs G such that G and its complement $\overline{G}$ are claw-free. With few exceptions, it is made of graphs whose connected components consist of cycles of length at least 4, paths, and of the complements of these graphs. Considering the hypergraph $\mathcal{H}^{(3)}(G)$ made of the 3-element subsets of the vertex set of a graph G on which G induces a clique or an independent subset, we deduce from above a description of the Boolean sum $G\dot{+}G'$ of two graphs G and G' giving the same hypergraph. We indicate the role of this latter description in a reconstruction problem of graphs up to complementation.

1. Results and motivation

Our notations and terminology mostly follow [3]. The graphs we consider in this paper are undirected, simple and have no loop. That is a *graph* is a pair $G := (V, \mathcal{E})$, where $\mathcal{E}$ is a subset of $[V]^2$, the set of 2-element subsets of V. Elements of V are the *vertices* of G and elements of $\mathcal{E}$ its *edges*. We denote by $V(G)$ the vertex set of G and by $E(G)$ its edge set. We look at members of $[V]^2$ as unordered pairs of distinct vertices. If A is a subset of V, the pair $G_{\restriction A} := (A, \mathcal{E} \cap [A]^2)$ is the *graph induced by G on A*. The *complement* of G is the simple graph $\overline{G}$ whose vertex set is V and whose edges are the unordered pairs of nonadjacent and distinct vertices of G, that is $\overline{G} = (V, \overline{\mathcal{E}})$, where $\overline{\mathcal{E}} = [V]^2 \setminus \mathcal{E}$. We denote by K_3 the complete graph on 3 vertices and by $K_{1,3}$ the graph made of a vertex linked to a $\overline{K_3}$. The graph $K_{1,3}$ is called a *claw*, the graph $\overline{K_{1,3}}$ a *co-claw*.

In [4], Brandstädt and Mahfud give a structural characterization of graphs with no claw and no co-claw; they deduce several algorithmic consequences (relying on bounded clique width). We will give a more precise characterization of such graphs. We denote by A_6 the graph on 6 vertices made of a K_3 bounded by three K_3 (cf. Figure 1) and by C_n the n-element cycle, $n \geq 4$. We denote by P_9 the Paley graph on 9 vertices (cf. Figure 1). Note that P_9 is isomorphic to its complement $\overline{P_9}$, to the line-graph of $K_{3,3}$ and also to $K_3 \square K_3$, the cartesian product of K_3 by itself (see [3] page 30 if needed for a definition of the *cartesian product of graphs*, and see [15] page 176 and [3] page 28 for a definition and basic properties of *Paley graphs*).

2000 *Mathematics Subject Classification.* 05C60; 05C99.

Key words and phrases. Graphs; claw-free graphs; cliques; independent subsets; Paley graphs.

Done under the auspices of the French-Tunisian CMCU "Outils mathématiques pour l'Informatique" 05S1505.

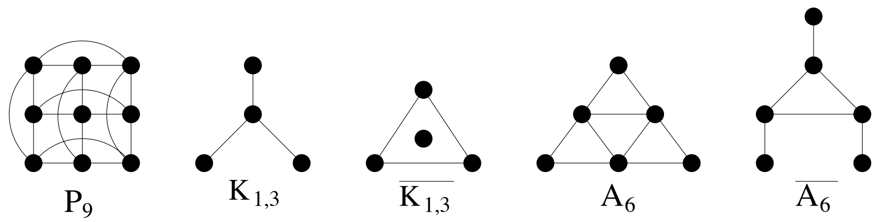

FIGURE 1

Given a set $\mathcal{F}$ of graphs, we denote by Forb$\mathcal{F}$ the class of graphs G such that no member of $\mathcal{F}$ is isomorphic to an induced subgraph of G. Members of Forb$\{K_3\}$, resp. Forb$\{K_{1,3}\}$ are called *triangle-free*, resp. *claw-free* graphs.

The main result of this note asserts:

Theorem 1.1. *The class* Forb$\{K_{1,3}, \overline{K_{1,3}}\}$ *consists of* A_6*, of the induced subgraphs of* P_9*, of graphs whose connected components consist of cycles of length at least* 4*, paths, and of the complements of these graphs.*

As an immediate consequence of Theorem 1.1, note that the graphs A_6 and $\overline{A_6}$ are the only members of Forb$\{K_{1,3}, \overline{K_{1,3}}\}$ which contain a K_3 and a $\overline{K_3}$ with no vertex in common. Note also that A_6 and $\overline{A_6}$ are very important graphs for the study of how maximal cliques and stable sets overlap in general graphs. See the main theorem of [7], see also [8]. Also, in [10], page 31, a list of all self-complementary line-graphs is given. A part from C_5, there are all induced subgraphs of P_9.

From Theorem 1.1 we obtain a characterization of the Boolean sum of two graphs having the same 3-homogeneous subsets. For that, we say that a subset of vertices of a graph G is *homogeneous* if it is a clique or an independent set (note that the word homogeneous is used with this meaning in Ramsey theory; in other areas of graph theory it has other meanings, several in fact). Let $\mathcal{H}^{(3)}(G)$ be the hypergraph having the same vertices as G and whose hyperedges are the 3-element homogeneous subsets of G. Given two graphs G and G' on the same vertex set V, we recall that the *Boolean sum* $G\dot{+}G'$ of G and G' is the graph on V whose edges are unordered pairs e of distinct vertices such that $e \in E(G)$ if and only if $e \notin E(G')$. Note that $E(G\dot{+}G')$ is the symmetric difference $E(G)\Delta E(G')$ of $E(G)$ and $E(G')$. The graph $G\dot{+}G'$ is also called the *symmetric difference* of G and G' and denoted by $G\Delta G'$ in [3]. Given a graph U with vertex set V, the *edge-graph* of U is the graph $S(U)$ whose vertices are the edges u of U and whose edges are unordered pairs uv such that $u = xy$, $v = xz$ for three distinct elements $x, y, z \in V$ such that yz is not an edge of U. Note that the edge-graph $S(U)$ is a spanning subgraph of $L(U)$, the *line-graph* of U, not to be confused with it.

Claw-free graphs and triangle-free graphs are related by means of the edge-graph construction. Indeed, as it is immediate to see, for every graph U, we have:

$$U \in Forb\{K_{1,3}\} \Longleftrightarrow S(U) \in Forb\{K_3\} \qquad (\star)$$

Our characterization is this:

Theorem 1.2. *Let U be a graph. The following properties are equivalent:*

(1) There are two graphs G and G' having the same 3-element homogeneous subsets such that $U := G\dot{+}G'$;
(2) $S(U)$ and $S(\overline{U})$ are bipartite;
(3) Either (i) U is an induced subgraph of P_9, or (ii) the connected components of U, or of its complement $\overline{U}$, are cycles of even length or paths.

As a consequence, if the graph U satisfying Property (1) is disconnected, then U contains no 3-element cycle, moreover, if U contains no 3-element cycle then each connected component of U is a cycle of even length, or a path, in particular U is bipartite.

The implication (2) $\Rightarrow$ (3) in Theorem 1.2 follows immediately from Theorem 1.1. Indeed, suppose that Property (2) holds, that is $S(U)$ and $S(\overline{U})$ are bipartite, then from Formula ($\star$) and from the fact that $S(A_6)$ and $S(C_n)$, $n \geq 4$, are respectively isomorphic to C_9 and to C_n, we have:

$$U \in Forb\{K_{1,3}, \overline{K_{1,3}}, A_6, \overline{A_6}, C_{2n+1}, \overline{C_{2n+1}} : n \geq 2\}.$$

From Theorem 1.1, Property (3) holds. The other implications, obtained by more straigthforward arguments, are given in Subsection 2.3.

This leaves open the following:

Problem 1.3. *Which pairs of graphs G and G' with the same 3-element homogeneous subsets have a given Boolean sum $U := G\dot{+}G'$?*

A partial answer, motivated by the reconstruction problem discussed below, is given in [6]. We mention that two graphs G and G' as above are determined by the graphs induced on the connected components of $U := G\dot{+}G'$ and on a system of distinct representatives of these connected components (Proposition 10 [6]).

A quite natural problem, related to the study of Ramsey numbers for triples, is this:

Problem 1.4. *Which hypergraphs are of the form $\mathcal{H}^{(3)}(G)$?*

An asymptotic lower bound of the size of $\mathcal{H}^{(3)}(G)$ in terms of $|V(G)|$ was established by A.W. Goodman [9].

The motivation for Theorem 1.2 (and thus Theorem 1.1) originates in a reconstruction problem on graphs that we present now. Considering two graphs G and G' on the same set V of vertices, we say that G and G' are *isomorphic up to complementation* if G' is isomorphic to G or to the complement $\overline{G}$ of G. Let k be a non-negative integer, we say that G and G' are *k-hypomorphic up to complementation* if for every k-element subset K of V, the graphs $G_{\restriction K}$ and $G'_{\restriction K}$ induced by G and G' on K are isomorphic up to complementation. Finally, we say that G is *k-reconstructible up to complementation* if every graph G' which is k-hypomorphic to G up to complementation is in fact isomorphic to G up to complementation. The following problem emerged from a question of P.Ille [12]:

Problem 1.5. *For which pairs (k, v) of integers, $k < v$, every graph G on v vertices is k-reconstructible up to complementation?*

It is immediate to see that if the conclusion of the problem above is positive for some k, v, then v is distinct from 3 and 4 and, with a little bit of thought, that if $v \geq 5$ then $k \geq 4$ (see Proposition 4.1 of [5]). With J. Dammak, G. Lopez [5] and [6] we proved that the conclusion is positive if:

(i) $4 \leq k \leq v-3$ or
(ii) $4 \leq k = v-2$ and $v \equiv 2$ $(mod\ 4)$.

We do not know if in (ii) the condition $v \equiv 2$ $(mod\ 4)$ can be dropped. For $4 \leq k = v-1$, we checked that the conclusion holds if $v = 5$ and noticed that for larger values of v it could be negative or extremely hard to obtain, indeed, a positive conclusion would imply that Ulam's reconstruction conjecture holds (see Proposition 19 of [6]).

The reason for which Theorem 1.2 plays a role in that matter relies on the following result, an easy consequence of the famous Gottlieb-Kantor theorem on incidence matrices ([11, 14]).

Proposition 1.6. *(Proposition 2.4 [5]) Let $t \leq min(k, v-k)$ and G and G' be two graphs on the same set V of v vertices. If G and G' are k-hypomorphic up to complementation then they are t-hypomorphic up to complementation.*

Indeed, if $3 \leq k \leq v-3$, Proposition 1.6 tells us that two graphs G and G' which are k-hypomorphic up to complementation are 3-hypomorphic up to complementation, which amounts to the fact that G and G' have the same 3-homogeneous subsets. A carefull study of such pairs G, G' allows to deal with the case $v = k+3$ and $v \equiv 1$ $mod\ 4$ (see [6]). Other cases use properties of the rank of some incidence matrices; notably a result of R.M. Wilson [16] on incidence matrices (and also the Gottlieb-Kantor theorem). In these cases the conclusion is stronger: G' or $\overline{G'}$ is equal to G (see [5]).

2. Proofs

Let U be a graph. For an unordered pair $e := xy$ of distinct vertices, we set $U(e) = 1$ if $e \in E(U)$ and $U(e) = 0$ otherwise. Let $x \in V(U)$; we denote by $N_U(x)$ and $d_U(x)$ the *neighborhood* and the *degree* of x (that is $N_U(x) := \{y \in V(U) : xy \in E(U)\}$ and $d_U(x) := |N_U(x)|$). For $X \subseteq V(U)$, we set $N_U(X) := (\cup_{x \in X} N_U(x)) \setminus X$.

2.1. **Proof of Theorem 1.1.** Trivially, the graphs described in Theorem 1.1 belong to $\mathrm{Forb}\{K_{1,3}, \overline{K_{1,3}}\}$. We prove the converse.

The *diamond* is the graph on four vertices with five edges. We say that *a graph G contains a graph H* when G has an induced subgraph isomorphic to H.

Theorem 2.1. *(Harary and Holzmann [13]) A graph G is the line-graph of a triangle-free graph if and only if G contains no claw and no diamond.*

Proof. Since [13] is very difficult to find, we include a short proof. Checking that a line-graph of a triangle-free graph contains no claw and no diamond is a routine matter. Conversely, let G be graph with no claw and no diamond. A theorem of Beineke [1] states that there exists a list $\mathcal{L}$ of nine graphs such any graph that does not contain a graph from $\mathcal{L}$ is a line-graph. One of the nine graphs is the claw and the eight remaining ones all contain a diamond. So, $G = L(R)$ for some graph R. Let R' be the graph obtained from R by replacing each connected component of R isomorphic to a triangle by a claw. So, $L(R) = L(R') = G$. We claim that R' is triangle-free. Else let T be a triangle of R'. From the construction of R', there is a vertex $v \notin T$ in the connected component of R' that contains T. So we may choose v with a neighbor in T. Now the edges of T and one edge from v to T induce a diamond of G, a contradiction. □

Let G be in the class $\mathrm{Forb}\{K_{1,3}, \overline{K_{1,3}}\}$.

(1) *We may assume that G and $\overline{G}$ are connected.*
Else, up to symmetry, G is disconnected. If G contains a vertex v of degree at least 3, then $N_G(v)$ contains an edge (for otherwise there is a claw), so G contains a triangle. This is a contradiction since by picking a vertex in another component we obtain a co-claw. So all vertices of G are of degree at most 2. It follows that the components of G are cycles (of length at least 4, or there is a co-claw) or paths, an outcome of the theorem. This proves (1).

(2) *We may assume that G contains no induced path P_6.*
Else G has an induced subgraph H that is either a path on at least 6 vertices or a cycle on at least 7 vertices. Suppose H maximal with respect to this property. If $G = H$, an outcome of the theorem is satisfied. Else, by (1), we pick a vertex v in $G \setminus H$ with at least a neighbor in H. From the maximality of H, v has a neighbor p_i in the interior of some $P_6 = p_1p_2p_3p_4p_5p_6$ of H. Up to symmetry we assume that v has a neighbor p_i where $i \in \{2, 3\}$. So $N_G(v) \cap \{p_1, p_2, p_3, p_4\}$ contains an edge e for otherwise $\{p_i, p_{i-1}, p_{i+1}, v\}$ induces a claw. If $e = p_1p_2$ then v must be adjacent to p_4, p_5, p_6 for otherwise there is a co-claw; so $\{v, p_1, p_4, p_6\}$ induces a claw. If $e = p_2p_3$ then v must be adjacent to p_5, p_6 for otherwise there is a co-claw, so from the symmetry between $\{p_1, p_2\}$ and $\{p_5, p_6\}$ we may rely on the previous case. If $e = p_3p_4$ then v must be adjacent to p_1, p_6 for otherwise there is a co-claw; so $\{v, p_1, p_3, p_6\}$ induces a claw. In all cases there is a contradiction. This proves (2).

(3) *We may assume that G contains no A_6 and no $\overline{A_6}$.*
Else up to a complementation, let aa', bb', cc' be three disjoint edges of G such that the only edges between them are ab, bc, ca. If $V(G) = \{a, a', b, b', c, c'\}$, an outcome of the theorem is satisfied, so let v be a seventh vertex of G. We may assume that $av \in E(G)$ (else there is a co-claw). If $a'v \in E(G)$ then $vb', vc' \in E(G)$ (else there is a co-claw) so $\{v, a', b', c'\}$ is a claw. Hence $a'v \notin E(G)$. We have $vb \in E(G)$ (or $\{a, a', v, b\}$ is a claw) and similarly $vc \in E(G)$. So $\{a', v, b, c\}$ is a co-claw. This proves (3).

(4) *We may assume that G contains no diamond.*
We prefer to think about this in the complement, so suppose for a contradiction that G contains a co-diamond, that is four vertices a, b, c, d that induce only one edge, say ab. By (1), there is a path P from $\{c, d\}$ to some vertex w that has a neighbor in $\{a, b\}$. We choose such a path P minimal and we assume up to symmetry that the path is from c.

If w is adjacent to both a, b then $\{a, b, w, d\}$ induces a co-claw unless w is adjacent to d, similarly w is adjacent to c, so $\{w, a, c, d\}$ induces a claw. Hence w is adjacent to exactly one of a, b, say to a. So, $P' = cPwab$ is an induced path and for convenience we rename its vertices $p_1, \cdots, p_k$. If d has a neighbor in P' then, from the minimality of P', this neighbor must be p_2. So, $\{p_2, p_1, p_3, d\}$ induces a claw. Hence, d has no neighbor in P'.

By (1), there is a path Q from d to some vertex v that has a neighbor in P'. We choose Q minimal with respect to this property. From the paragraph above, $v \neq d$. Let p_i (resp. p_j) be the neighor of v in P' with minimum (resp. maximum) index. If $i = j = 1$ then $dQvp_1Pwp_{k-1}p_k$ is a path on at least 6 vertices a contradiction to (2). So, if $i = j$ then $i \neq 1$ and symmetrically, $i \neq k$, so $\{p_{i-1}, p_i, p_{i+1}, v\}$ is a claw.

Hence $i \neq j$. If $j > i+1$ then $\{v, v', p_i, p_j\}$, where v' is the neighbor of v along Q, is a claw. So, $j = i+1$. So vp_ip_j is a triangle. Hence $P' = p_1p_2p_3p_4$, $Q = dv$ and $i = 2$, for otherwise there is a co-claw. Hence, $P' \cup Q$ form an induced $\overline{A_6}$ of G, a contradiction to (3). This proves (4).

Now G is connected and contains no claw and no diamond. So, by Theorem 2.1, G is the line-graph of some connected triangle-free graph R. Symmetrically, $\overline{G}$ is also a line-graph. These graphs are studied in [2].

If R contains a vertex v of degree at least 4 then all edges of R must be incident to v, for else an edge e non-incident to v together with three edges of R incident to v and non-incident to e form a co-claw in G. So all vertices of R have degree at most 3 since otherwise, G is a clique, a contradiction to (1). We may assume that R has a vertex a of degree 3 for otherwise G is a path or a cycle. Let b, b', b'' be the neighbors of a. Since a has degree 3, all edges of R must be incident to b, b' or b'' for otherwise G contains a co-claw.

If one of b, b', b'', say b, is of degree 3, then $N_R(b) = \{a, a', a''\}$ and all edges of R are incident to one of a, a', a'' (or there is a co-claw). So R is a subgraph of $K_{3,3}$. So, since $P_9 = L(K_{3,3})$, $G = L(R)$ is an induced subgraph of P_9, an outcome of the theorem. Hence we assume that b, b', b'' are of degree at most 2. If $|N_R(\{b, b', b''\}) \setminus \{a\}| \geq 3$, then R contains the pairwise non-incident edge $bc, b'c', b''c''$ say, and the edges $ab, ab', ab'', bc, b'c', b''c''$ are vertices of G that induce an $\overline{A_6}$, a contradiction to (3). So, $|N_R(\{b, b', b''\}) \setminus \{a\}| \leq 2$ which means again that R is a subgraph of $K_{3,3}$. $\square$

2.2. **Ingredients for the proof of Theorem 1.2.** The proof of the equivalence between Properties (1) and (2) of Theorem 1.2 relies on the following lemma.

Lemma 2.2. *Let G and G' be two graphs on the same vertex set V and let $U := G \dot{+} G'$. Then, the following properties are equivalent:*

(a) G and G' have the same 3-element homogeneous subsets;

(b) $U(\{x,y\}) = U(\{x,z\}) \neq U(\{y,z\}) \Longrightarrow G(\{x,y\}) \neq G(\{x,z\})$ for all distinct elements x, y, z of V.

(c) The sets $A_1 := E(U) \cap E(G)$ and $A_2 := E(U) \setminus E(G)$ divide $V(S(U))$ into two independent sets and also the sets $B_1 := E(\overline{U}) \cap E(G)$ and $B_2 := E(\overline{U}) \setminus E(G)$ divide $V(S(\overline{U}))$ into two independent sets.

Proof. Observe first that Property (b) is equivalent to the conjunction of the following properties:

(b_U): If uv is an edge of $S(U)$ then $u \in E(G)$ iff $v \notin E(G)$.

and

$(b_{\overline{U}})$: If uv is an edge of $S(\overline{U})$ then $u \in E(G)$ iff $v \notin E(G)$.

$(a) \Longrightarrow (b)$. Let us show $(a) \Longrightarrow (b_U)$.

Let $uv \in E(S(U))$, then $u, v \in E(U)$. By contradiction, we may suppose that $u, v \in E(G)$ (the other case implies $u, v \in E(G')$ thus is similar). Since u and v are edges of $U = G \dot{+} G'$ then $u, v \notin E(G')$. Let $w := yz$ such that $u = xy$, $v = xz$. Then $w \notin E(U)$ and thus $w \in E(G)$ iff $w \in E(G')$.

If $w \in E(G)$, $\{x, y, z\}$ is an homogeneous subset of G. Since G and G' have the same 3-element homogeneous subsets, $\{x, y, z\}$ is an homogeneous subset of G'. Hence, since $u, v \notin E(G')$, $w = yz \notin E(G')$, thus $w \notin E(G)$, a contradiction.

If $w \notin E(G)$, then $w \notin E(G')$; since $u, v \notin E(G')$ it follows that $\{x, y, z\}$ is an homogeneous subset of G'. Consequently $\{x, y, z\}$ is an homogeneous subset of G.

Since $u, v \in E(G)$, then $w \in E(G)$, a contradiction.
The implication $a) \Longrightarrow (b_{\overline{U}})$ is similar.
$(b) \Longrightarrow (a)$. Let T be a K_3 of G. Suppose that T is not an homogeneous subset of G' then we may suppose $T = \{u, v, w\}$ with $u, v \in E(G')$ and $w \notin E(G')$ or $u, v \in E(\overline{G'})$ and $w \notin E(\overline{G'})$. In the first case $uv \in E(S(\overline{U}))$, which contradicts Property $(b_{\overline{U}})$, in the second case $uv \in E(S(U))$, which contradicts Property (b_U).
$(b) \Longrightarrow (c)$. First $V(S(U)) = E(U) = A_1 \cup A_2$ and $V(S(\overline{U})) = E(\overline{U}) = B_1 \cup B_2$. Let u, v be two distinct elements of A_1 (respectively A_2). Then $u, v \in E(G)$ (respectively $u, v \notin E(G)$). From (b_U) we have $uv \notin E(S(U))$. Then A_1 and A_2 are independent sets of $V(S(U))$. The proof that B_1 and B_2 are independent sets of $V(S(\overline{U}))$ is similar.
$(c) \Longrightarrow (b)$. This implication is trivial. □

2.3. **Proof of Theorem 1.2.** Implication (1) $\Longrightarrow$ (2) follows directly from implication $(a) \Longrightarrow (c)$ of Lemma 2.2. Indeed, Property (c) implies trivially that $S(U)$ and $S(\overline{U})$ are bipartite.
(2) $\Longrightarrow$ (1). Suppose that $S(U)$ and $S(\overline{U})$ are bipartite. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be respectively a partition of $V(S(U)) = E(U)$ and $V(S(\overline{U})) = E(\overline{U})$ into independent sets. Note that $A_i \cap B_j = \emptyset$, for $i, j \in \{1, 2\}$. Let G, G' be two graphs with the same vertex set as U such that $E(G) = A_1 \cup B_1$ and $E(G') = A_2 \cup B_1$. Clearly $E(G \dot{+} G') = A_1 \cup A_2 = E(U)$. Thus $U = G \dot{+} G'$. To conclude that Property (1) holds, it suffices to show that G and G' have the same 3-element homogeneous subsets, that is Property (a) of Lemma 2.2 holds. For that, note that $A_1 = E(U) \cap E(G)$, $A_2 = E(U) \setminus E(G)$, $B_1 = E(\overline{U}) \cap E(G)$ and $B_2 = E(\overline{U}) \setminus E(G)$ and thus Property (c) of Lemma 2.2 holds. It follows that Property (a) of this lemma holds.

The proof of implication (2) $\Longrightarrow$ (3) was given in Section 1. For the converse implication, let U be a graph satisfying Property (3). It is clear from Figure 1 that $S(P_9)$ is bipartite (vertical edges and horizontal edges form a partition). Since $\overline{P_9}$ is isomorphic to P_9, $S(\overline{P_9})$ is bipartite too. Thus, if U is isomorphic to an induced subgraph of P_9, Property (2) holds. If not, we may suppose that the connected components of U are cycles of even length, paths (otherwise, replace U by $\overline{U}$). In this case, $S(U)$ is trivially bipartite. In order to prove that Property 2 holds, it suffices to prove that $S(\overline{U})$ is bipartite too. This is a direct consequence of the following claim:

Claim 2.3. *If U is a bipartite graph, then $S(\overline{U})$ is bipartite too.*

Proof. If $c : V(U) \to \mathbb{Z}/2\mathbb{Z}$ is a colouring of U, set $c' : V(S(\overline{U})) \to \mathbb{Z}/2\mathbb{Z}$ defined by $c'(\{x, y\}) := c(x) + c(y)$. □

With this, the proof of Theorem 1.2 is complete.

2.4. **A direct proof for (3) $\Longrightarrow$ (1) of Theorem 1.2.** In [6] we gave all possible decompositions of U. When $U = P_9$, a decomposition $U = G \dot{+} G'$ can be given by a picture (see figure 2).

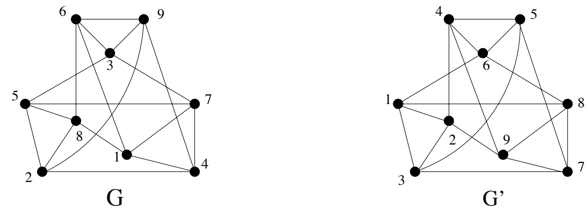

FIGURE 2

Let $n \geq 2$. Let X_n be an n-element set, $x_0, \cdots, x_{n-1}$ be an enumeration of X_n, $X_n^0 := \{x_i \in X_n : i \equiv 0 \ (mod\ 2)\}$ and $X_n^1 := X_n \setminus X_n^0$. Set $R_n := [X_n^1]^2 \cup [X_n^2]^2$, $S_n := \{\{x_{2i}, x_{2i+1}\} : 2i < n\}$, $S'_n := \{\{x_{2i+1}, x_{2i+2}\} : 2i < n-1\}$. Let M_n and M'_n be the graphs with vertex set X_n and edge sets $E(M_n) := R_n \cup S_n$ and $E(M'_n) := R_n \cup S'_n$ respectively. Let $M''_n := (X_n, R_n \cup S'_n \cup \{\{x_0, x_{n-1}\}\})$ for n even, $n \geq 4$. For $n \in \{6, 7\}$ we give a picture (see figure 3). For convenience, we set $M_1 = M'_1$ the graph with one vertex and we put $V(M_1) := X_1^0 := \{x_0\}$. When G is a graph of the form M_n, M'_n, or M''_n, with $n \geq 1$, we put $V^0(G) := X_n^0$ and $V^1(G) := X_n^1$.

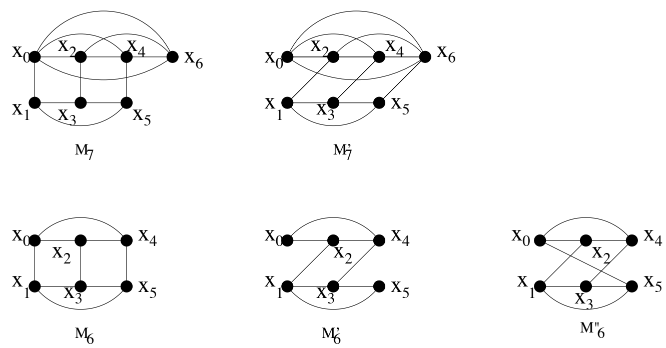

FIGURE 3

When U is a cycle of even size $2n$, a decomposition $U = G \dot{+} G'$ can be given by $G = M_{2n}$ and $G' = M''_{2n}$. When U is a path of size n, a decomposition $U = G \dot{+} G'$ can be given by $G = M_n$ and $G' = M'_n$.

When the connected components of U are cycles of even length or paths, we define G and G' satisfying $U = G \dot{+} G'$ as follows: For each connected component C of U, (G_C, G'_C) is given by the previous step. For distinct connected components C and C' of U, $x \in C$, $x' \in C'$, $xx' \in E(G)$ (and $xx' \in E(G')$) if and only if $x \in V^0(G_C)$ and $x' \in V^0(G_{C'})$, or $x \in V^1(G_C)$ and $x' \in V^1(G_{C'})$.

When the connected components of $\overline{U}$ are cycles of even length or paths, from $\overline{U} = \overline{G} \dot{+} G'$, the previous step gives a pair $(\overline{G}, G')$, then a pair (G, G').

Acknowledgements

We thank S. Thomassé for his helpful comments.

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